Finite Element Method – Mixed Formulation

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Finite Element Method

- Original problem:

\[
\begin{cases}
- \nabla \cdot K \nabla u = f & \text{in } \Omega \\
u = 0 & \text{in } \Gamma
\end{cases}
\]

- **Weak formulation** (variational problem)

Find \( u \in H^1_g(\Omega) \) such that

\[
\int_{\Omega} \nabla u \cdot K \nabla \omega \, dx = \int_{\Omega} f \omega \, dx, \quad \forall \omega \in H^1_0(\Omega)
\]

- Approximated solution \( u_h \in H^1_g(\Omega_h) \) is represented as a linear combination of basic functions:

\[
u_h = \sum_{i=1}^{n_h} \alpha_i \varphi_i(x).
\]

- Linear approximation P1 – elementwise linear functions

Any function in \( H^1_g(\Omega_h) \) is uniquely defined by its values in nodes of discretization.
- **Construction of basis in** $H^1_h(\Omega_h)$

Basis function $\varphi_i$, connected to a node $i$ of a triangulation $T_h$ is:

- continuous function $\Omega_h \to R$
- linear on element

- for any $\varphi_i$, $\varphi_i(\bar{x}_j) = \delta_{ij} = \begin{cases} 1 & , \quad i = j \\ 0 & , \quad i \neq j, \; \bar{x}_j - \text{node.} \end{cases}$

\[ \sum_{i=1}^{n} \varphi_i = 1 \; \text{for} \; \forall \bar{x} \in \Omega_h \]
Substituting, we get a system of linear equations \( A \alpha = F \) with unknowns \( \alpha_i, i = 1, \ldots, n_i \).

Matrix elements:
\[
A_{ij} = \int_{\Omega} \nabla \varphi_i(x) \cdot K \nabla \varphi_j(x) \, dx, \quad i, j = 1, \ldots, n_i
\]

RHS vector:
\[
F_i = \int_{\Omega} f(x) \varphi_i(x) \, dx
\]

- A is sparse.

Piecewise constant derivatives, what may cause problems while solving transport equation, tracing streamlines, etc., (mass conservation does not hold within an element).

Mixed formulation brings a remedy
Finite elements – Mixed formulation

• Original problem: Find $u : \Omega \rightarrow R$ such that:

\begin{align*}
- \nabla \cdot K \nabla u &= f \quad \text{in} \quad \Omega \\
\nabla u |_{\partial \Omega} &= 0
\end{align*}

(1)

• Apply an equivalent system of equations:

\begin{align*}
q &= -K \nabla u \\
\nabla \cdot q &= f \\
u |_{\partial \Omega} &= 0
\end{align*}

(2)

\begin{align*}
K^{-1} q + \nabla u &= 0 \\
\nabla \cdot q &= f \\
u |_{\partial \Omega} &= 0
\end{align*}

(3)

Two unknown functions: $u$ (pressure, head), $q$ (velocity, volumetric flux)
Standard procedure towards variational formulation:
First equation of (3) is multiplied by a vector function \( \mathbf{v} \in H^1_{0N}(\text{div}; \Omega) \), while the second by a scalar function \( \varphi \in L^2(\Omega) \) and all is integrated over \( \Omega \).

\[
\begin{cases}
\int K^{-1} \mathbf{q} \cdot \mathbf{v} + \int \nabla u \cdot \mathbf{v} = 0 \\
\int \nabla \cdot \mathbf{q} \cdot \varphi = \int f \cdot \varphi \\
u|_{\partial \Omega} = 0
\end{cases}
\]

(4)

Green’s Theorem is applied to the second term of the first equation:

\[
\int_{\Omega} \nabla u \cdot \mathbf{v} = -\int_{\Omega} u \cdot \nabla \cdot \mathbf{v} + \int_{\partial \Omega} u \cdot \mathbf{n} \cdot \mathbf{v}
\]
Weak formulation in the Mixed form:
Find \((q, u) \in H^{1}_{0,N}(\text{div}; \Omega) \times L^{2}(\Omega)\) such that

\[
\begin{aligned}
\int_{\Omega} K^{-1} q \cdot v - \int_{\Omega} u \cdot \nabla \cdot v &= 0 \quad \forall \ v \in H^{1}_{0,N}(\text{div}; \Omega) \\
-\int_{\Omega} \nabla \cdot q \cdot \varphi &= -\int_{\Omega} f \cdot \varphi \quad \forall \varphi \in L^{2}(\Omega)
\end{aligned}
\]

(5)

Under appropriate assumptions, there exists a unique solution of the problem.

Solutions belong to spaces which should be consistent.
Condition Ładyżenska - Brezzi – Babuška.
Approximate representations:

\[(6a) \quad q_h = \sum_{i=1}^{m} \alpha_i \psi_i\]

\(q_h\) and \(\psi_i\) are vectorial functions, coefficients \(\alpha_i\) are scalars.

Degrees of freedom related to \(\psi_i\) are linked to edges (2D), or faces (3D) of elements.

\[(6b) \quad u_h = \sum_{i=1}^{n} \beta_i \varphi_i\]

Degrees of freedom \(\beta_i\) are linked to elements.

Substituting (6) to (5):

\[
\left\{
\begin{array}{l}
\sum_{i=1}^{m} \alpha_i \int_{\Omega} \left( (K^{-1} \psi_i) \cdot \psi_j \right) - \sum_{i=1}^{n} \beta_i \int_{\Omega} \nabla \cdot \psi_j = 0 \\
- \sum_{i=1}^{m} \alpha_i \int_{\Omega} \nabla \cdot \psi_i \varphi_k = - \int_{\Omega} f \varphi_k
\end{array}
\right.
\]

\[(7) \quad j=1,..., m \text{ (edges)}, k=1,...,n \text{ (elements)}\]
It is a system of linear equations with \( m+n \) unknowns:

\[
\begin{bmatrix}
A & B \\
B^T & 0
\end{bmatrix}
\begin{bmatrix}
\alpha \\
\beta
\end{bmatrix} =
\begin{bmatrix}
F_1 \\
F_2
\end{bmatrix},
\]

where:

\[
A = \int_{\Omega} \left( K^{-1} \psi_i \right)^T \psi_j, \\
B = -\int_{\Omega} \nabla \cdot \psi_i \varphi_j
\]

\( F_1 = 0 \) (here nonzero Dirichlet BC appears)

Remark: Neumann BC is strongly fulfilled in the Mixed formulation.
Dirichlet BC is fulfilled in a weak sense (a natural BC).

\( F_2 = -\int_{\Omega} f \varphi_j \)

We need spaces now.

1) $u$ can be approximated by **elementwise constant functions** (no continuity across elements is needed):

$$V_h = \{ v|_K \text{ constant} \}$$

2) Approximation of $H^1_0(\Omega; \mathcal{N})$ is more complicated.

If $q$ is smooth on an element $\left( q|_K \in H^1(K) \right)$ then

$$\int_{\Omega} |\nabla \cdot q|^2 < \infty$$

if and only if $n.q$ is **continuous** across elements.
Degrees of freedom are normal components of $\mathbf{q}$ in some points on faces / (edges in 2D).

- **Construction of basis in $H^1_h(\Omega_h)$**

Basis function $\varphi_i$, connected to a node $i$ of a triangulation $T_h$ is:
- continuous function $\Omega_h \rightarrow \mathbb{R}$
- linear on element

for any $\varphi_i, \varphi_i(\mathbf{x}_j) = \delta_{ij} = \begin{cases} 1 , & i = j \\ 0 , & i \neq j \end{cases}$, $\mathbf{x}_j$ – node.

Let $e_i, i=1,...,m$ denote numbered faces (edges in 2D), and $K_j, j=1,...,n$ denote numbered elements.

Our vectorial space is spanned by linearly independent vectorial basis functions $\psi_i$, such that for any $\psi_i$

$$\int_{e_j} \mathbf{n}_j \cdot \psi_i \, d\gamma = \delta_{ij} = \begin{cases} 1 , & i = j \\ 0 , & i \neq j \end{cases}$$

for $i, j=1,...,M$.

Any function $\mathbf{q}$ belonging to this space has one degree of freedom per face / edge,

$$\int_{e_j} \mathbf{n}_i \cdot \mathbf{q} \, d\gamma \ (\text{flux through } e_i).$$
Construction of functions fulfilling:

\[
\int_{e_j} \mathbf{n}_j \cdot \psi_i \, d\gamma = \delta_{ij} \quad \int_{e_j} \nabla \cdot \psi_i \, d\gamma = 1
\]

An “easy-triangle” example:

\[
\psi_j = \frac{1}{2\Delta_T} \begin{bmatrix} x - x_2 \\ y - y_2 \end{bmatrix}
\]

\[
2\Delta_T = 1
\]
Example in 1D

Not very convincing...

\[- \frac{d^2 u}{dx^2} = 10 \, w(0, 0.5)\]

\[u(0) = 0, \quad \frac{du}{dx}\bigg|_0 = 0\]

Exact solution: \(u(x) = -5x^2 + 5x\).

We take:

- elementwise functions to approximate \(u\) (pressure),
- elementwise linear functions to approximate \(q\) (velocity).
Finite elements – Mixed formulation in 1D

<table>
<thead>
<tr>
<th>FLUX</th>
<th>FUNCTION</th>
<th>VALUE</th>
</tr>
</thead>
<tbody>
<tr>
<td>0,041667</td>
<td>0,020833</td>
<td>0</td>
</tr>
<tr>
<td>0,020833</td>
<td>0,083333</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>0,020833</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>0,083333</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Elementwise constant function

5
3,75
2,5
1,25
0,286458
0,755208
1,067708
1,223958
Towards hybridization

System of linear equations (8) is not good.
- Matrix, though symmetric, is not positive-defined.
- Large size of the matrix (number of elements + number of faces/edges).

In order to obtain a hybrid version:
- Continuity requirements on faces/edges are relaxed. This way the space of approximate solutions for velocities $q$ is enlarged.
- In order to enforce continuity a new variable is introduced acting on faces/edges (Lagrange multipliers).

Each face/edge is ‘considered’ twice.
What happens:

\[ \int_{\Omega} K^{-1} q \cdot v + \int_{\Omega} \nabla u \cdot v = 0 \]

Green’s theorem is applied for each element independently.

\[ \int_{\Omega} \nabla u \cdot v = \sum_{K \in T_h} \int_{K} \nabla u \cdot v = \sum_{K \in T_h} \left( -\int_{K} u \nabla \cdot v + \int_{\partial K} u \mathbf{n} \cdot v \right) \]

In order to enforce continuity of \( v \):

\[ \sum_{K \in T_h \cap \partial K} \int_{K} \mathbf{n} \cdot v \mu = 0 \quad \forall \mu \text{ constant on } \partial e, \]

Plus boundary conditions:

\[ \sum_{K \in T_h \cap \partial K} \int_{K} \mathbf{n} \cdot v \mu \, d\gamma = \int_{\partial \Omega} g N \mu \, d\gamma \quad \forall \mu \in \ldots \]
**Mixed-Hybrid formulation:**

Find \((q_h, u_h, \lambda_h) \in \ldots\) such that:

\[
\int_{\Omega} (K^{-1} q_h) \cdot v_h - \sum_{K \in T_h} \left( \int_K u_h \nabla \cdot v_h - \int_{\partial K} \lambda_h \mathbf{n} \cdot v_h \right) = 0 \quad \forall v_h \ldots
\]

\[
- \sum_{K \in T_h} \left( \int_K \nabla \cdot v_h \varphi_h \right) = -\int_{\Omega} f \varphi_h \quad \forall \varphi_h \ldots
\]

\[
\sum_{K \in T_h \partial K} \int_K \mathbf{n} \cdot v_h \mu_h \, d\gamma = \int_{\partial \Omega} g_N \mu_h \, d\gamma \quad \forall \mu_h \in \ldots
\]

\(v_h\) vector function defined before

\(\varphi_h\) constant on elements \(K\)

\(\mu_h\) constant on faces/edges, a ‘trace’ of \(u\).

There exists a unique solution of this system. Moreover, this solution are the same as the solution of the problem in mixed formulation.

**Remark:** in mixed-hybrid formulation, Dirichlet BCs are strongly fulfilled. Neumann BC are fulfilled in a weak sense.
We arrived to a (hudge!) system of linear equations:

$$
\begin{bmatrix}
A & B & C \\
B^T & 0 & 0 \\
C^T & 0 & 0
\end{bmatrix}
\begin{array}{c}
\alpha \\
\beta \\
\lambda
\end{array} =
\begin{array}{c}
F_1 \\
F_2 \\
F_3
\end{array}
$$

So what have we gained ?????????????????????

- It is possible to compute \( \alpha \) elementwise:
  \[
  \alpha = A^{-1}(F_1 - B\beta - C\lambda)
  \]

- After substitution and some algebra:
  \[
  \beta = (B^T A^{-1} B)^{-1}(B^T A^{-1}(F_1 - C\lambda) - F_2)
  \]

- To finally get:
  \[
  D\lambda = F, \text{ with } D = \ldots \text{ and } F = \ldots \text{ D is a matrix 3x3 in 2D (and 4x4 in 3D).}
  \]

This ‘small’ (local) system of equations is inserted into a global matrix, using a global numbering of faces/edges.

- The size of the final system of linear equations: number of faces/edges
- Once \( \lambda \) is computed, \( \alpha \) and \( \beta \) are computed elementwise.
- Mass conservation is perfectly fulfilled
infiltration from a waste disposal: 0.004 m/d, Regional flow 0.002 m/d
Domain 200m x 60 m homogeneous, isotropic

Conductivity coefficients: 1 m/day (blue), 0.01 m/day (red)